

REGULAR SEQUENCES AND THE JOINT SPECTRAL RADIUS

MICHAEL COONS

ABSTRACT. We classify the growth of a k -regular sequence based on information from its k -kernel. In order to provide such a classification, we introduce the notion of a growth exponent for k -regular sequences and show that this exponent is equal to the joint spectral radius of any set of a special class of matrices determined by the k -kernel.

1. INTRODUCTION

Let \mathbb{K} be a field of characteristic zero. The k -kernel of $f : \mathbb{Z}_{\geq 0} \rightarrow \mathbb{K}$ is the set

$$\text{Ker}_k(f) := \{f(k^\ell n + r)\}_{n \geq 0 : \ell \geq 0, 0 \leq r < k^\ell}.$$

The sequence f is called k -automatic provided the set $\text{Ker}_k(f)$ is finite [6]. In 1992, as a generalisation of automatic sequences, Allouche and Shallit [1] introduced the notion of regular sequences. By their definition, a sequence f taking values in \mathbb{K} is called k -regular, for an integer $k \geq 1$, provided the \mathbb{K} -vector space $\langle \text{Ker}_k(f) \rangle_{\mathbb{K}}$ spanned by $\text{Ker}_k(f)$ is finite dimensional. Connecting regular sequences to finite sets of matrices, Allouche and Shallit [1, Lemma 4.1] showed that a \mathbb{K} -valued sequence f is k -regular if and only if there exist a positive integer d , a finite set of matrices $\mathcal{A}_f = \{\mathbf{A}_0, \dots, \mathbf{A}_{k-1}\} \subseteq \mathbb{K}^{d \times d}$, and vectors $\mathbf{v}, \mathbf{w} \in \mathbb{K}^d$ such that $f(n) = \mathbf{w}^T \mathbf{A}_{i_0} \cdots \mathbf{A}_{i_s} \mathbf{v}$, where $(n)_k = i_s \cdots i_0$ is the base- k expansion of n . Moreover, their proof showed that all such collections of matrices can be described (or constructed) by considering spanning sets of $\langle \text{Ker}_k(f) \rangle_{\mathbb{K}}$.

In their seminal paper, Allouche and Shallit [1, Theorem 2.10] proved that given a k -regular sequence f , there is a positive constant c_f such that $f(n) = O(n^{c_f})$.

In this paper, we determine the optimal value of the constant c_f . To state our result, we require a few definitions. Let $k \geq 1$ be an integer and $f : \mathbb{Z}_{\geq 0} \rightarrow \mathbb{K}$ be a (not eventually zero) k -regular sequence. We define the *growth exponent* of f , denoted $\text{GrExp}(f)$, by

$$\text{GrExp}(f) := \limsup_{\substack{n \rightarrow \infty \\ f(n) \neq 0}} \frac{\log |f(n)|}{\log n}.$$

The *joint spectral radius* of a finite set of matrices $\mathcal{A} = \{\mathbf{A}_0, \mathbf{A}_1, \dots, \mathbf{A}_{k-1}\}$, denoted $\rho(\mathcal{A})$, is defined as the real number

$$\rho(\mathcal{A}) = \limsup_{n \rightarrow \infty} \max_{0 \leq i_0, i_1, \dots, i_{n-1} \leq k-1} \|\mathbf{A}_{i_0} \mathbf{A}_{i_1} \cdots \mathbf{A}_{i_{n-1}}\|^{1/n},$$

where $\|\cdot\|$ is any (submultiplicative) matrix norm. This quantity was introduced by Rota and Strang [8] and has a wide range of applications. For an extensive treatment, see Jungers's monograph [7].

Theorem 1. *Let $k \geq 1$ and $d \geq 1$ be integers and $f : \mathbb{Z}_{\geq 0} \rightarrow \mathbb{K}$ be a (not eventually zero) k -regular sequence. If \mathcal{A}_f is any collection of k integer matrices associated to a basis of the \mathbb{K} -vector space $\langle \text{Ker}_k(f) \rangle_{\mathbb{K}}$, then*

$$\log_k \rho(\mathcal{A}_f) = \text{GrExp}(f),$$

where \log_k denotes the base- k logarithm.

Date: November 25, 2015.

The research of M. Coons was supported by ARC grant DE140100223.

We note that Theorem 1 holds for \mathbb{K} replaced by any Nötherian ring R , where \mathcal{A}_f is any collection of k matrices associated to an R -module basis of the R -module spanned by $\text{Ker}_k(f)$, where this R -module is viewed as an R -submodule of the set of a sequences with entries in R . In particular, the result holds for the ring \mathbb{Z} .

Remark 2. In engineering circles, for certain choices of \mathcal{A} related to a set D of forbidden sign patterns, the quantity $\log_2 \rho(\mathcal{A})$ is sometimes referred to as the *capacity* of the set D , denoted $\text{cap}(D)$. See Jungers, Blondel, and Protasov [4, Section II] for details.

2. THE GROWTH EXPONENT OF A REGULAR SEQUENCE

In this section, all matrices are assumed to have entries in \mathbb{K} and all regular sequences are supposed to not eventually be zero.

Lemma 3. *Let $k \geq 1$ be an integer and $\mathcal{A} = \{\mathbf{A}_0, \mathbf{A}_1, \dots, \mathbf{A}_{k-1}\}$ be a finite set of matrices. Given $\varepsilon > 0$ then there is a submultiplicative matrix norm $\|\cdot\|$ such that $\|\mathbf{A}_i\| < \rho(\mathcal{A}) + \varepsilon$ for each $i \in \{0, 1, \dots, k-1\}$.*

Lemma 3 can be found in Blondel et al. [5, Proposition 4], though it was first given in the original paper of Rota and Strang [8].

Proposition 4. *Let $k \geq 2$ be an integer and $f : \mathbb{Z}_{\geq 0} \rightarrow \mathbb{K}$ be a k -regular function. For any $\varepsilon > 0$, there is a constant $c = c(\varepsilon) > 0$ such that for all $n \geq 1$,*

$$\frac{|f(n)|}{n^{\log_k(\rho(\mathcal{A}_f) + \varepsilon)}} \leq c,$$

where \mathcal{A}_f is the set any set of matrices associated to a spanning set of $\langle \text{Ker}_k(f) \rangle_{\mathbb{K}}$.

Proof. Let $\varepsilon > 0$ be given and let $\|\cdot\|$ be a matrix norm such that the conclusion of Lemma 3 holds. Then

$$|f(n)| \leq \|\mathbf{v}\| \cdot \|\mathbf{w}\| \cdot \prod_{j=0}^s \|\mathbf{A}_{i_j}\| \leq \|\mathbf{v}\| \cdot \|\mathbf{w}\| \cdot (\rho(\mathcal{A}) + \varepsilon)^s,$$

where the base- k expansion of n is $i_s \cdots i_0$. Using the bound $s \leq \log_k n$ with some rearrangement gives the result. \square

Lemma 5. *Let $k \geq 1$ be an integer and $\mathcal{A} = \{\mathbf{A}_0, \mathbf{A}_1, \dots, \mathbf{A}_{k-1}\}$ be a finite set of matrices. If $\varepsilon > 0$ is a real number, then there is a positive integer m and a matrix $\mathbf{A}_{i_0} \cdots \mathbf{A}_{i_{m-1}}$, such that*

$$(\rho(\mathcal{A}) - \varepsilon)^m < \rho(\mathbf{A}_{i_0} \cdots \mathbf{A}_{i_{m-1}}) < (\rho(\mathcal{A}) + \varepsilon)^m.$$

Proof. This is a direct consequence of the definition of the joint spectral radius. \square

Now let $k \geq 2$ be an integer, and suppose that $f : \mathbb{Z}_{\geq 0} \rightarrow \mathbb{K}$ is an unbounded k -regular sequence. Given a word $w = i_s \cdots i_0 \in \{0, \dots, k-1\}^*$, we let $[w]_k$ denote the natural number such that $(n)_k = w$. Let $\{\{f(n)\}_{n \geq 0} = \{g_1(n)\}_{n \geq 0}, \dots, \{g_d(n)\}_{n \geq 0}\}$ be a basis for the \mathbb{K} -vector space $\langle \text{Ker}_k(f) \rangle_{\mathbb{K}}$. Then for each $i \in \{0, 1, \dots, k-1\}$, the sequences $\{g_1(kn + i)\}_{n \geq 0}, \dots, \{g_d(kn + i)\}_{n \geq 0}$ can be expressed as \mathbb{K} -linear combinations of $\{g_1(n)\}_{n \geq 0}, \dots, \{g_d(n)\}_{n \geq 0}$ and hence there is a set of $d \times d$ matrices $\mathcal{A}_f = \{\mathbf{A}_0, \dots, \mathbf{A}_{k-1}\}$ with entries in \mathbb{K} such that

$$\mathbf{A}_i [g_1(n), \dots, g_d(n)]^T = [g_1(kn + i), \dots, g_d(kn + i)]^T$$

for $i = 0, \dots, k-1$ and all $n \geq 0$. In particular, if $i_s \cdots i_0$ is the base- k expansion of n , then $\mathbf{A}_{i_0} \cdots \mathbf{A}_{i_s} [g_1(0), \dots, g_d(0)]^T = [g_1(n), \dots, g_d(n)]^T$. (We note that this holds even if we pad the base- k expansion of n with zeros at the beginning.) We call such a set of matrices \mathcal{A}_f , constructed in this way, a *set of matrices associated to a basis of $\langle \text{Ker}_k(f) \rangle_{\mathbb{K}}$* .

This construction allows us to provide a lower bound analogue of Proposition 4.

Proposition 6. *Let $k \geq 2$ be an integer and $f : \mathbb{Z}_{\geq 0} \rightarrow \mathbb{K}$ be a k -regular function. For any $\varepsilon > 0$, there is a constant $c = c(\varepsilon) > 0$ such that for infinitely many $n \geq 1$,*

$$\frac{|f(n)|}{n^{\log_k(\rho(\mathcal{A}_f) - \varepsilon)}} \geq c,$$

where \mathcal{A}_f is any set of matrices associated to a basis of $\langle \text{Ker}_k(f) \rangle_{\mathbb{K}}$.

Proof. Let $\varepsilon > 0$ be given. Then by Lemma 5 there is a positive integer m and a matrix $\mathbf{A} = \mathbf{A}_{i_0} \cdots \mathbf{A}_{i_{m-1}}$ such that $\rho(\mathbf{A}) > (\rho(\mathcal{A}_f) - \varepsilon)^m$. Let λ be an eigenvalue of \mathbf{A} with $|\lambda| = \rho(\mathbf{A})$. Then there is an eigenvector \mathbf{y} such that $\mathbf{A}\mathbf{y} = \lambda\mathbf{y}$. Pick a vector \mathbf{x} such that $\mathbf{x}^T \mathbf{y} = c_1 \neq 0$. Then

$$|\mathbf{x}^T \mathbf{A}^n \mathbf{y}| = |c_1| \cdot |\lambda|^n = |c_1| \cdot \rho(\mathbf{A})^n > |c_1| \cdot (\rho(\mathcal{A}_f) - \varepsilon)^{nm}.$$

Using a method developed by Bell, Coons, and Hare [3], it follows (see Appendix A for details) that there are words $u_1, \dots, u_d, v_1, \dots, v_t$ from $\{0, 1, \dots, k-1\}^*$ and a positive constant c_2 such that for each $n \geq 0$ there is an element from

$$\{|f([u_i(i_{m-1} \cdots i_0)^n v_j]_k)| : i = 1, \dots, d, j = 1, \dots, t\},$$

which is at least $c_2(\rho(\mathcal{A}_f) - \varepsilon)^{nm}$. Here, as previously, we have used the notation $[w]_k$ to be the integer n such that $(n)_k = w$.

If $M = \max\{|u_i|, |v_j| : i = 1, \dots, d, j = 1, \dots, t\}$, then

$$N = [u_i(i_{m-1} \cdots i_0)^n v_j]_k < k^{2M+nm},$$

so that $\log_k(N) - 2M < nm$. Thus, by the finding of the previous paragraph, there are infinitely many N such that

$$\frac{|f(N)|}{N^{\log_k(\rho(\mathcal{A}_f) - \varepsilon)}} = \frac{|f(N)|}{(\rho(\mathcal{A}_f) - \varepsilon)^{\log_k N}} > \frac{c_2}{(\rho(\mathcal{A}_f) - \varepsilon)^{2M}},$$

which is the desired result. \square

Proof of Theorem 1. For a given $\varepsilon > 0$, Proposition 4 implies that

$$\lim_{n \rightarrow \infty} \frac{|f(n)|}{n^{\log_k(\rho(\mathcal{A}_f) + 2\varepsilon)}} = 0,$$

and Proposition 6 implies that

$$\limsup_{n \rightarrow \infty} \frac{|f(n)|}{n^{\log_k(\rho(\mathcal{A}_f) - 2\varepsilon)}} = \infty.$$

Taken together these give

$$\log_k(\rho(\mathcal{A}_f) - 2\varepsilon) \leq \text{GrExp}(f) \leq \log_k(\rho(\mathcal{A}_f) + 2\varepsilon).$$

Since ε can be taken arbitrarily small, this proves the theorem. \square

We end this section by highlighting one major difference between Proposition 4 and Proposition 6. Proposition 4 is true for \mathcal{A}_f related to any spanning set of the \mathbb{K} -vector space $\langle \text{Ker}_k(f) \rangle_{\mathbb{K}}$, while Proposition 6 requires \mathcal{A}_f to be associated to a basis of $\langle \text{Ker}_k(f) \rangle_{\mathbb{K}}$. In fact, these two propositions give the following corollary.

Corollary 7. *Let $k \geq 2$ be an integer and $f : \mathbb{Z}_{\geq 0} \rightarrow \mathbb{K}$ be a k -regular function. If \mathcal{B}_f is any set of matrices associated to f and \mathcal{A}_f is any set of matrices associated to a basis of $\langle \text{Ker}_k(f) \rangle_{\mathbb{K}}$, then $\rho(\mathcal{A}_f) \leq \rho(\mathcal{B}_f)$.*

Equality in the conclusion of the above corollary would be desirable, but unfortunately, this is not (in general) the case. To see this, consider the 2-regular function f , where, for $(n)_2 = i_s \cdots i_0$, we have $f(n) = \mathbf{w}^T \mathbf{A}_{i_0} \cdots \mathbf{A}_{i_s} \mathbf{v}$, with

$$\mathcal{A}_f = \{\mathbf{A}_0, \mathbf{A}_1\} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}, \quad \mathbf{w}^T = [1 \ 0], \quad \text{and} \quad \mathbf{v} = [1 \ 1]^T.$$

Then also for any number $x > 1$, we have $f(n) = \mathbf{x}^T \mathbf{B}_{i_0} \cdots \mathbf{B}_{i_s} \mathbf{y}$, with

$$\mathcal{B}_f = \{\mathbf{B}_0, \mathbf{B}_1\} = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & x \end{pmatrix} \right\}, \quad \mathbf{x}^T = [1 \ 0 \ 0], \quad \text{and} \quad \mathbf{y} = [1 \ 1 \ 0]^T,$$

and

$$\rho(\mathcal{A}_f) = 1 < x = \rho(\mathcal{B}_f).$$

APPENDIX A.

For a given $\varepsilon > 0$, we had by Lemma 5 that there is a positive integer m and a matrix $\mathbf{A} = \mathbf{A}_{i_0} \cdots \mathbf{A}_{i_{m-1}}$ such that $\rho(\mathbf{A}) > (\rho(\mathcal{A}_f) - \varepsilon)^m$. Choosing an eigenvalue λ of \mathbf{A} with $|\lambda| = \rho(\mathbf{A})$, we found vectors \mathbf{x} and \mathbf{y} such that $\mathbf{x}^T \mathbf{y} = c_1 \neq 0$ and

$$(1) \quad |\mathbf{x}^T \mathbf{A}^n \mathbf{y}| = |c_1| \cdot |\lambda|^n = |c_1| \cdot \rho(\mathbf{A})^n > |c_1| \cdot (\rho(\mathcal{A}_f) - \varepsilon)^{nm}.$$

In this appendix, we follow an argument of Bell, Coons, and Hare [3, p. 198] to provide the existence of words $u_1, \dots, u_d, v_1, \dots, v_t$ from $\{0, 1, \dots, k-1\}^*$ such that for each $n \geq 0$ there is an element from

$$\{ |f([u_i(i_{m-1} \cdots i_0)^n v_j]_k)| : i = 1, \dots, d, j = 1, \dots, t \},$$

which is at least $c_2(\rho(\mathcal{A}_f) - \varepsilon)^{nm}$.

To this end, let $k \geq 2$ be an integer, suppose that $f : \mathbb{Z}_{\geq 0} \rightarrow \mathbb{K}$ is an unbounded k -regular sequence, and $\mathcal{A}_f = \{\mathbf{A}_0, \dots, \mathbf{A}_{k-1}\}$ be a set of matrices associated to a basis $\{f(n)\}_{n \geq 0} = \{g_1(n)\}_{n \geq 0}, \dots, \{g_d(n)\}_{n \geq 0}\}$ of the \mathbb{K} -vector space $\langle \text{Ker}_k(f) \rangle_{\mathbb{K}}$.

We claim that the \mathbb{K} -span of the vectors $[g_1(i), \dots, g_d(i)]$, as i ranges over all natural numbers, must span all of \mathbb{K}^d . If this were not the case, then their span would be a proper subspace of \mathbb{K}^d and hence the span would have a non-trivial orthogonal complement. In particular, there would exist $c_1, \dots, c_d \in \mathbb{K}$, not all zero, such that

$$c_1 g_1(n) + \cdots + c_d g_d(n) = 0$$

for every n , contradicting the fact that $g_1(n), \dots, g_d(n)$ are \mathbb{K} -linearly independent sequences.

Let $\langle \mathcal{A}_f \rangle$ denote the semigroup generated by the elements of \mathcal{A}_f . We have just shown that there exist words $\mathbf{X}_1, \dots, \mathbf{X}_d$ in $\langle \mathcal{A}_f \rangle$ such that

$$[g_1(0), \dots, g_d(0)] \mathbf{X}_1, \dots, [g_1(0), \dots, g_d(0)] \mathbf{X}_d$$

span \mathbb{K}^d .

Now consider $\mathbf{x}^T \mathbf{A}^n \mathbf{y}$ as described in the first paragraph of this appendix. By construction, we may write $\mathbf{x}^T = \sum_j \alpha_j [g_1(0), \dots, g_d(0)] \mathbf{X}_j$ for some complex numbers α_j . Then

$$\mathbf{x}^T \mathbf{A}^n = \sum_j \alpha_j [g_1(0), \dots, g_d(0)] \mathbf{X}_j \mathbf{A}^n.$$

Let u_j be the word in $\{0, 1, \dots, k-1\}^*$ corresponding to \mathbf{X}_j and let $y = i_s \cdots i_0$ be the word in $\{0, \dots, k-1\}^*$ corresponding to \mathbf{A} ; that is $y = i_s \cdots i_0$ where $\mathbf{A} = \mathbf{A}_{i_s} \cdots \mathbf{A}_{i_0}$ and similarly for u_j . Then we have

$$[g_1(0), \dots, g_d(0)] \mathbf{X}_j \mathbf{A}^n = [g_1([u_j y^n]_k), \dots, g_d([u_j y^n]_k)]^T.$$

Write $\mathbf{y}^T = [\beta_1, \dots, \beta_d]$. Then

$$\mathbf{x}^T \mathbf{A}^n \mathbf{y} = \sum_{i,j} \alpha_i \beta_j g_j([u_i y^n]_k).$$

By assumption, each of $\{g_1(n)\}_{n \geq 0}, \dots, \{g_d(n)\}_{n \geq 0}$ is in the \mathbb{K} -vector space generated by $\text{Ker}_k(f)$, and hence there exist natural numbers p_1, \dots, p_t and q_1, \dots, q_t with $0 \leq q_m < k^{p_m}$ for $m = 1, \dots, t$ such that each of for $s = 1, \dots, d$, we have $g_s(n) = \sum_{i=1}^t \gamma_{i,s} f(k^{p_i} n + q_i)$ for some constants $\gamma_{i,s} \in \mathbb{K}$. Then

$$\mathbf{x}^T \mathbf{A}^n \mathbf{y} = \sum_{i,j,\ell} \alpha_i \beta_j \gamma_{\ell,j} f([u_i y^n v_\ell]_k),$$

where v_ℓ is the unique word in $\{0, 1, \dots, k-1\}^*$ of length p_ℓ such that $[v_\ell]_k = q_\ell$. Let $K = \sum_{i,j,\ell} |\alpha_i| \cdot |\beta_j| \cdot |\gamma_{\ell,j}|$. Then since $|\mathbf{x}^T \mathbf{A}^n \mathbf{y}| \geq |c_1| \cdot (\rho(\mathcal{A}_f) - \varepsilon)^{nm}$ for all n , some element from

$$\{|f([u_i y^n v_j]_k)| : i = 1, \dots, d, j = 1, \dots, t\}$$

is at least $(|c_1|/K) \cdot (\rho(\mathcal{A}_f) - \varepsilon)^{nm}$ for each n .

Acknowledgements. We thank Björn Rüffer for several useful conversations.

REFERENCES

1. Jean-Paul Allouche and Jeffrey Shallit, *The ring of k -regular sequences*, Theoret. Comput. Sci. **98** (1992), no. 2, 163–197. MR 1166363 (94c:11021)
2. Jason P. Bell, Michael Coons, and Kevin G. Hare, *Growth degree classification for finitely generated semigroups of integer matrices*, Semigroup Forum, to appear.
3. ———, *The minimal growth of a k -regular sequence*, Bull. Aust. Math. Soc. **90** (2014), no. 2, 195–203. MR 3252000
4. Vincent D. Blondel, Raphaël Jungers, and Vladimir Protasov, *On the complexity of computing the capacity of codes that avoid forbidden difference patterns*, IEEE Trans. Inform. Theory **52** (2006), no. 11, 5122–5127. MR 2300380 (2007m:94086)
5. Vincent D. Blondel, Yurii Nesterov, and Jacques Theys, *On the accuracy of the ellipsoid norm approximation of the joint spectral radius*, Linear Algebra Appl. **394** (2005), 91–107. MR 2100578 (2005i:15043)
6. Alan Cobham, *Uniform tag sequences*, Math. Systems Theory **6** (1972), 164–192. MR 0457011 (56 #15230)
7. Raphaël Jungers, *The joint spectral radius*, Lecture Notes in Control and Information Sciences, vol. 385, Springer-Verlag, Berlin, 2009, Theory and applications. MR 2507938 (2011c:15001)
8. Gian-Carlo Rota and Gilbert Strang, *A note on the joint spectral radius*, Nederl. Akad. Wetensch. Proc. Ser. A 63 = Indag. Math. **22** (1960), 379–381. MR 0147922 (26 #5434)

SCHOOL OF MATH. AND PHYS. SCIENCES, UNIVERSITY OF NEWCASTLE, CALLAGHAN, AUSTRALIA
E-mail address: Michael.Coons@newcastle.edu.au